CHAPTER 2
LIMITS AND CONTINUITY
INTRODUCTION TO CHAPTER 3
DERIVATIVES

2.1. RATES OF CHANGE AND LIMITS

A. We will save rates of change for later.
B. The official definition of a limit is a tedious beast. It arrived much later than the concept.
C. The concept of a limit: If you can get as close as you want to (without actually getting there), you are essentially there.
D. Notation: $\lim_{x \to c}$ f(x) = L means that the y-coordinate is heading toward L on f(x) as x approaches c from both directions.
E. Graphical considerations to consider with regards to limits:

1. If both directions do not approach the same y-value, we say that the limit does not exist.
2. If the y value shoots off to $\pm$ infinity then we say that the limit is plus or minus ∞. (Technically, infinity has no limit).
3. The function does not have to behave at x = c for the limit to exist. It is all about the approaching.

F. It is possible to just consider a one-sided limit:

1. $\lim_{x \to c^+}$ f(x) approaches from the right.
2. $\lim_{x \to c^-}$ f(x) approaches from the left.

G. Note that for $\lim_{x \to c}$ f(x) to exist, then $\lim_{x \to c^+}$ f(x) = $\lim_{x \to c^-}$ f(x)
H. On piece-wise defined functions we usually look at the edges of the definition.
I. Assignment: P. 62 {1 – 6, 31, 32 (interval notation on h and i), 45 - 48 (Instructions on p. 63)}

J. Methods for finding limits: NAG

1. Look at the graph. (Graphically)
2. Make a chart from both sides. (Numerically)
3. Analytically.

K. Speaking of analytically, let us look at a few theorems:

1. $\lim_{x \to c} k = k$ (the limit on a horizontal line remains constant)
2. $\lim_{x \to c} x = c$ (the y value is equal to the x value on y = x)
3. The limit of a ________ is the ________ of the limit(s).
   Fill in the blanks with:
   a. sum b. difference c. product d. quotient
   e. power f. root g. constant multiple

L. A summary of the limit theorems is: *If you can plug it in, plug it in.*

M. If you plug in a limit and get the indeterminate $\frac{0}{0}$, you can try to factor, reduce and plug in again.

N. Using the concept of the sandwich theorem, here are two special limits that you should know:

$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0$

O. On most of the trig limits, just look at the graph and skip the analytical.

P. The TI-89 knows how to take limits

Q. Assignment: P.62 {7 - 15 odd, 18, 20 - 26, 43, 44}

2.2. LIMITS INVOLVING INFINITY

A. The end behavior of a function is the y value of the function the far left and right sides.

B. If the end behavior flattens out then we have a horizontal asymptote.

C. Notationally, y = b is a horizontal asymptote of the graph of a function y = f(x) if either

$\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

D. Recall your rules for horizontal asymptotes of rational functions that you learned in Pre-Calculus: Check the degree on the top (T) and the bottom (B):

I) if T>B, then f(x) diverges to either plus or minus infinity.
II) if T=B, then the horizontal asymptote is the ratio of the leading coefficients.
III) if T<B, then the horizontal asymptote is zero.

E. The calculator graph is our friend if we extend x-min and x-max.

F. It is possible for a function to have more than one horizontal asymptote. Examples include certain square roots that lead to absolute values.

G. Note that the two ends of a function are often different. Consider $y = 2^{(-x)}$, and $y = x^3$.

H. The books definition of a Vertical Asymptote: When the one sided limit shoots off to plus or minus infinity.

I. Our approach to a V.A.: When you try to evaluate and you get a number over zero. We attack by considering the sign (to stick on ∞) of a value close to each side of the vertical asymptote.

J. Finding power function end behavior models for polynomials and rational functions.

K. Create a graph from certain limit information.

L. Assignment: P. 71 {3 – 6, 10 - 16 even (graphs or tables), 17 - 21, 29 - 38, 49, and 50}

2.3. CONTINUITY

A. The concept of a continuous function.
B. The definition of continuity at a point

> 1. For an interior point: If the limit from the left is the limit from the right is the value of the function then you have continuality.
> 2. For an end point: It must go where you think that it should go, yet from only one direction.

C. If a function is not continuous at a point, then it is discontinuous at the point. There are various names for various types of discontinuities as described on page 76.
D. We can use a piece-wise defined function to fill in a hole of a removable discontinuity (by extending the function).
E. We can also figure out what values a piece-wise defined function must take on in order to be continuous at a point. Make sure that both sides agree according to the definition of continuity.
F. The definition of continuity on an interval: If it is continuous at every point on the interval. Note that a function may be continuous on one interval, while not on a different interval.
G. Where should we look for discontinuities? In denominators, under even roots, within piece-wise defined functions.
H. Assignment: P. 80 {2 - 10 even, 11 – 16, 20 – 24 even, 35 - 38, and 41}

I. Our first named theorem: The Intermediate Value Theorem. If f(x) is a continuous function on [a, b], then for any y_0 between f(a) and f(b) there exists a "c" within [a, b] such that f(c) = y_0.
J. A picture of the IVT and examples that include finding zeros, (or any other value) of a function and falling objects.
K. Applying an instance of the IVT to a function on a given domain.
L. Note that the biggest problem for students with the IVT is starting with x-values instead of y-values.

2.4. RATES OF CHANGE AND TANGENT LINES

A. Definitions:

1. A secant line is a line that intersects a curve at more than one point.
2. The average rate of change is the slope of the secant line at the endpoints of an interval.
3. The instantaneous rate of change is the slope of a tangent line at a point.

B. To find the slope of a line tangent to a curve f(x) at x = a,

> 1. We start with the secant line between (a, f(a)) and (after going h units to the right) (a + h, f(a + h)). The slope of this line is:
> $m = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$ the last of which is called the difference quotient of f(x) at x = a.
> 2. To move from a secant line to the tangent line, we let h go to zero.
> m = $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ provided the limit exists.

C. The slope of the tangent does not exist where the curve is not "smooth".
D. For the tangent line to exist on piece-wise defined functions, the slope must be the same on both sides of a continuous function. These are found through two one-sided limits.
E. Definition: A *normal* line is perpendicular to the tangent line.
F. In terms of velocity (which is the rate of change of the position function)

1. The average velocity is the slope of the secant line.
2. The instantaneous velocity is the slope of the tangent line.

G. We can find the slope of the tangent line at a specific point or a generic point "a" if we desire to find the slope at more than one value.
H. Assignment: P.87 {2 - 12 even, 15, 16, 19, 21,and 23}

3.1. DERIVATIVE OF A FUNCTION

A. Definition: The derivative of the function f with respect to the variable x is the function f ' whose value at x is: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists.

B. An alternate definition of the derivative at a point x = a is: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$, provided the limit exists.

C. A function is said to be differentiable at a point if the derivative exists at that point. If the derivative exists at every point within the domain, the function is said to be a differentiable function. Graphically, for the most part, a differentiable function is "smooth".
D. There are many different notations for the derivative. We will use them all.
E. Assignment: P. 101 {1 – 6}

F. We can use our calculators to find the derivative of a function at a point. We can also use them to find the equation of the tangent line.
G. Graphing the derivative from a function or the graph of a function

1. If you want to sketch the graph of the derivative when given the graph of y = f(x):
   a. In general, you are plotting the slope of f(x) as the y-value of f '(x) for each value of x in the domain
   b. Plot a zero at each horizontal tangent.
   c. Note that the derivative of a polynomial is one degree less than the original, and the right hand behavior will be the same for both.
   d. Whenever f(x) is rising, f '(x) will be positive.
   e. Whenever f(x) is falling, f '(x) will be negative.
2. If you know the function, stick it in y_1. Let y_2 = NDER(y_1,x,x).

H. Derivatives from discrete data points:

1. Find the slope of two points closest to your point. (3 possibilities)
2. Estimate with the midpoint.

I. For one-sided derivatives we just let h approach zero from a particular direction.

J. Determining where piece-wise defined functions are differentiable (if the function is continuous and the one-sided limits are equal) and sketching their derivatives.

K. Assignment: P.101 {7 - 14 (use y_2 = NDER(y_1,x,x) on 13 and 14), 16, 18,
Estimate the derivative at t = 5 from the data given in problem 19}

3.2. DIFFERENTIABILITY

A. The derivative does not exist

1. Where the curve is not smooth - a corner or a cusp.
2. Where the curve is not continuous.
3. Where the curve has a vertical tangent. (The derivative introduces a zero in the denominator and therefore has a different domain than the function)

B. Theorem: If a function is differentiable at a point then it is continuous at the point.

C. Assignment: P. 111 {1 - 10}

Mandatory Review: P. 91 {1 – 3, 5 - 7, 15 - 25, 27 – 35 odd, 39, 41, 43, and 47},
P. 173 {55 - 57, 59}