

CHAPTER 4

APPLICATIONS OF DERIVATIVES

4.1. EXTREME VALUES OF FUNCTIONS

- A) Absolute (or global) extrema are the most extreme y values on the interval (domain)
- I) $f(c)$ is the minimum if $f(c) \leq f(x)$ for all x on the interval
 - II) $f(c)$ is the maximum if $f(c) \geq f(x)$ for all x on the interval
- B) The Extreme Value Theorem: If f is continuous on the closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval.
- C) Relative (or local) extrema occur within an open interval. They are the extreme y values in some small area. Officially, they cannot be the endpoints. Your book extends the concept to include the endpoints.
- D) Absolute extrema will occur at relative extrema or at closed endpoints.
- E) Finding absolute extrema on the graphs of closed intervals.
- F) Assignment: P. 184 {1 – 10}
- G) Defn: c is a critical number if $f'(c) = 0$ or $f'(c)$ is undefined while $f(c)$ is defined.
- H) A critical point would be $(c, f(c))$ where c is a critical number. (*It is smooth where the derivative of the top = 0 and pointy where the derivative of the bottom = 0 [where the point exists of course]*)
- I) All relative extrema occur at critical numbers, but all critical numbers do not produce relative extrema.
- J) Applying the Extreme Value Theorem: Absolute extrema exist either at endpoints or critical points of continuous functions. Find the values of the function at the endpoints and critical points and quickly determine the absolute extrema.
- K) Your calculator knows how to find absolute extrema! Unfortunately, you can only use this for confirmation after applying the theorems.
- L) Finding critical numbers on polynomial, exponential, rational and piece-wise defined functions.
- M) Assignment: P. 184 {11 - 13, 15, 38 - 44 even (just find the critical numbers), 45 - 48}

4.2. MEAN VALUE THEOREM

- A) M.V.T. for Derivatives: If f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists at least one c on (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- B) The M.V.T. finds where the instantaneous rate of change equals the average rate of change over an interval.
- C) The M.V.T. finds the point where the slope of the tangent line is parallel to the slope of the secant line.
- D) Assignment: P. 192 {15 - 18, 21 - 22, and 39}

- E) The M.V.T. can be used to prove the following theorem about increasing and decreasing intervals: If f is continuous on $[a, b]$ and differentiable on (a, b) and:
- I) $f'(x) > 0$ for all x on (a, b) , then f is increasing on $[a, b]$
 - II) $f'(x) < 0$ for all x on (a, b) , then f is decreasing on $[a, b]$
- F) Finding increasing and decreasing intervals (which are closed if the function is continuous at a given endpoint).
- i) Use critical numbers (and discontinuities) to establish intervals of increasing and decreasing
 - ii) Test the intervals by plugging in a value from that interval into the first derivative.
 - iii) Positive values indicate that the interval is increasing, negative is decreasing.
- G) There is a fancy number line shortcut for notational purposes.
- H) Sketching graphs when given information about the derivative on intervals and at points.
- D) Assignment: P. 192 {Just state the intervals of increasing and decreasing on 2 - 14 even, do 36 - 38}
- I) The book uses the M.V.T. for the introduction and leading to the definition of the *antiderivative*: A function $F(x)$ is an antiderivative of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f .
- J) There are an infinite number of antiderivatives of a function. They all differ by their constant. Some would call them a family.
- K) Solving a problem by getting the antiderivative is the same as solving a differential equation. When we are given an initial value, we can find the value of the constant.
- L) Assignment: P. 192 {25 - 34}

4.3. CONNECTING F' AND F'' WITH THE GRAPH OF F

- A) The first derivative test for relative extrema:
- i) If successive intervals change from increasing to decreasing, you have a relative max (Except if it is discontinuous)
 - ii) If successive intervals change from decreasing to increasing, you have a relative min (Except if it is discontinuous)
- B) Finding relative extrema for polynomials is gentle and easy.
- C) Finding relative extrema for rational expressions like $y = \frac{x-2}{x^2-1}$ are more difficult, but far more interesting.
- D) Assignment: P. 204 {8 - 28 even, minus 18 and 24, just parts a, b, and e} leave plenty of room to complete each of these problems for the next assignment.
- E) Concavity: On a differentiable open interval, f is concave up when f' is increasing, f is concave down when f' is decreasing.
- F) On an open interval, if $f''(x) > 0$, then the graph of f is concave up, if $f''(x) < 0$, then the graph of f is concave down.

- G) Test intervals of the second derivative to find intervals of concavity. Use values where the second derivative is zero or does not exist to establish your intervals. Always use open intervals for concavity.
- H) A point of inflection occurs at a change in concavity (at a differentiable, continuous point). The y value of the point is found in the original function.
- I) If $(c, f(c))$ is a point of inflection then either $f''(c) = 0$ or $f''(c)$ is undefined while $f'(c)$ is defined. (It is kind of like a critical point for the first derivative. And just like critical points, just because $f''(c) = 0$, there need not be a point of inflection. Consider $y = x^4$).
- J) Points of inflection in your life: Logistic Curves and the Normal Curve.
- K) Assignment: P. 204 {8 - 28 even, minus 18 and 24, do parts c, d, and f}
- L) The Second Derivative Test for Relative Extrema: If c is a critical number and
- i) $f''(c) > 0$, then $f(c)$ is a relative minimum
 - ii) $f''(c) < 0$, then $f(c)$ is a relative maximum
 - iii) $f''(c) = 0$ or does not exist, use the first derivative test
- M) The second derivative test is easy for polynomials. It is also easy to forget. This is one of those things that the A.P. folks like to sneak in and see who really knows their stuff.
- N) Note that the expression “twice differentiable” means that each of the first and the second derivatives exist at all points on a given interval.
- O) Fun with graphs:
- i) Given a graph, find intervals for the signs of f , f' , and f'' .
 - ii) Given a graph, find points on the graph satisfying the signs of f , f' , and f'' .
 - iii) Given the graph of f' , find intervals for the signs of f , f' , and f'' .
 - iv) Given the signs for intervals of f , f' , and f'' , come up with the graph
- P) Assignment: Assignment: P. 203 {2, 4, 33, 34, 41, 42, 45 - 48}.

4.4. MODELING AND OPTIMIZATION

- A) Optimization seeks a maximum or a minimum within an application.
- B) A Technique
- Write an equation for what you are trying to optimize. Note the domain.
 - Use a possible secondary equation to get it in terms of one variable on the right side.
 - Take the derivative, set it equal to zero, and solve. (Use critical numbers.)
 - Make sure it is the correct extrema. Are the endpoints applicable?
 - Answer the question.
- C) Examples include maximizing areas and minimizing fences
- D) Examples include maximizing profits and minimize costs. Profit = Revenue - Cost
- E) Examples include maximizing volume for various situations.
- F) Assignment: P. 214 {1, 7, 9, 10, (analytically only on 1 – 10), 11, 13, 25, and 49}
- G) Examples include optimizing inscribed geometric figures.
- H) The can problem minimizes materials.
- I) Examples include minimizing distances. (You can optimize a radical by optimizing the radicand.)

- J) Some of these problems are just plain difficult. Good luck.
- K) Assignment: P. 214 {4, 5, 6, 16, 20, 36, 37, and 42}

4.5. LINEARIZATION AND NEWTON'S METHOD

- A) Major Concept: If you zoom in far enough, almost every curve looks like a line.
- B) We can use the tangent line to find points close to a known point.
- C) The tangent line at $(a, f(a))$ is: $L(x) = f(a) + f'(a)(x - a)$.
- D) Linearization for implicit functions seems more reasonable than for explicit functions.
- E) Newton's Method finds the zeros or roots of a function. It is no longer in the A.P. curriculum because your calculator can find zeros very quickly.
- F) A differential uses the linearization. $dy = f'(x) dx$. It approximates the change in y .
- G) Sometimes we are interested in finding the difference between the actual y value and the estimated y value as determined by linearization.
- H) Assignment: P. 229 {Find the linearization function for 1 - 6; do 16, 20, 28}

4.6. RELATED RATES

- A) Related rates means that there is a relationship between two rates.
- B) A Technique that works (for the most part)
 - i) Write the correct equation
 - ii) Substitute (if necessary) to get it down to two variables
 - iii) Take the derivative with respect to time
 - iv) One of the rates will be given, plug it in
 - v) Solve for the remaining rate (this is the magic equation)
 - vi) Answer the question by plugging into the magic equation; you may have to work to get a value to plug in. Make sure your answer has correct units.
- C) The easier problems do not have a secondary equation that you will need to substitute.
- D) Examples include working with the formulas for the area of a circle, the volume of a sphere, and angles and/or lengths of a right triangle.
- E) The biggest mistake is plugging in a fixed value for a changing variable. Do the calculus on the changing variable and then plug in the fixed value that you are concerned about.
- F) Assignment: P. 237 {1, 2, 11, 13, 14, 15, 20, 21, and 29}
- G) It is possible to be given more than one rate. In that case, you do not have to get it down to two variables. An example would include problem #9.
- H) Most of the time when you are given three variables, you substitute a secondary equation to get down to two variables. Examples of this include the cone problem.
- I) The most difficult aspect of many of these problems is sorting out the equation that relates all of the variables. Ex. 36
- J) Assignment: P. 237 {10, 12, 16, 22, 24, 32, 34, 36 (which requires the law of cosines)}

Mandatory Review: P. 242 {1, 5, 8, 13, 15, 24, 27, 32, 36, 47, 54, 57, 58, 60, and 62}
 1997 BC Multiple Choice: {3, 8, 12, 23, 80}