CHAPTER 5
THE DEFINITE INTEGRAL

5.1. ESTIMATING WITH FINITE SUMS

A) A geographical reason why the area under a curve can be useful. Area = length * width Distance = Velocity * Time. Note how the area is related to the units.

B) When the area is not a nice rectangle, we can approximate the area by breaking it up into many rectangles. This can be done in three ways: LRAM, RRAM, and MRAM.

C) We should be able to estimate the area from a function, table of values or a graph.

D) To make our job easier for known functions we have a calculator program.

E) The more rectangles, the better the approximation of the area. When enough rectangles are used, all three area measures will be close to one another in value.

F) Given discrete points we can only use LRAM and RRAM as in problem #11. Given a continuous function we can find the midpoints and use MRAM as in problem #9.

G) Assignment: P.254 {1 - 8, 10, 12, and 26}

5.2. DEFINITE INTEGRALS

A) If we let the number of rectangles go to infinity, and if the continuous function is above the x-axis on a given interval [a, b], then we know that A = $\underset{n \to \infty}{Lim} \sum_{k=1}^{n} f(c_k)\Delta x$

(where c_k is some x value on the kth interval and $\Delta x = \frac{b-a}{n}$, the width of the rectangle)

B) While the Area is represented by the above limit only when the curve is at or above the x-axis, the limit exits for any continuous function, (although the limit does not represent the area when the curve falls below the x-axis). This limit represents the definite integral of the function over the interval and has its own special notation:

$$\underset{n \to \infty}{Lim} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b \boldsymbol{f(x)dx}$$

C) The official Riemann Sum notation is a little different. It is more general, but with the same result: $\underset{\|P\| \to 0}{Lim} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b \boldsymbol{f(x)dx}$

D) We will start our investigation of the definite integral by finding the area under curves that have shapes with known geometric formulas.

E) A little theorem: if c is a constant: $\int_a^b c\,dx$ = c(b-a).

F) Assignment: P. 267 {2 – 26 even}

G) Areas and the definite integral:

i) If the function is above the x-axis on [a, b] then this number represents the area.

ii) If the function is below the x-axis on [a, b] then the opposite of this number represents the area.

iii) If the function ever goes below the x-axis (or, in general), we can go backwards and evaluate the definite integral using areas:

$$\int_a^b f(x)dx = \text{(area above the x-axis) - (area below the x-axis).}$$

H) We can find the definite integral on the calculator using NINT (or fnint), a function integrator. The TI-89 has two options for evaluating the definite integral. If $\int_a^b f(x)dx$ is too ugly, you are advised to go straight to NINT.

I) All continuous functions are integrable. Some discontinuous functions are integrable. As long as infinity is not involved (as in a vertical asymptote), then the function is integrable. We can therefore find the definite integral of piece-wise defined functions.

J) An investigation into the definite integral using summation formulas.

K) Assignment: P. 268 {39 - 43, 45, and 48}

5.3. DEFINITE INTEGRALS AND ANTIDERIVATIVES

A) A few definite integral theorems:

I) If a function is continuous on [a, b], then it is integrable on [a, b]

II) $\int_a^a f(x)dx = 0$

III) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

IV) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

V) $\int_a^b k * f(x)dx = k * \int_a^b f(x)dx$

VI) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

VII) There are no theorems for products or quotients.

B) Manipulating the theorems for a given picture

C) The Average Value of a Function on an Interval: If f is integrable on [a, b], then the average value of f on the interval is $\frac{1}{b-a}\int_a^b f(x)dx$

D) The Mean Value Theorem for Integrals: if f is continuous on [a, b], then there exists a c within [a, b] such that f(c) = $\frac{1}{b-a}\int_a^b f(x)dx$. It is the x-coordinate of the value of y, which is the average value of the function.

E) A pictorial representation of the M.V.T. for integrals requires just the smallest bit of algebra: $\int_a^b f(x)dx$ = f(c)(b-a) In other words, you can make a rectangle out of the area under a curve, (if the curve is above the x-axis on the interval). The y-coordinate of where the rectangle intersects the curve is the average value of the function.

F) Find the average y-value of any function. Find the average velocity in two different ways.

G) Assignment: P. 274 {1 - 6; you can use calculator justification on 21 - 28}

5.4. FUNDAMENTAL THEOREM OF CALCULUS

A) One version of the F.T.C. (Which your book calls part two): If F (x) is the antiderivative of f (x) and f (x) is continuous on [a, b], then $\int_a^b f(x)dx$ = F(b) - F(a).

Another version of this is: $\int_a^b f'(x)dx = f(b) - f(a)$

B) The F.T.C. relates derivatives (in this case, antiderivatives), with areas! The area under a curve can be found using the ideas that gave us the slope of the curve. The derivative and the integral are inverse operations. One takes a step down the function ladder (derivatives) and one takes a step up the function ladder (definite integral).

C) Correct notation for using the F.T.C.

D) The biggest problem will be figuring out the anti-derivative. We will concentrate on this in the next chapter, but if you know your derivative formulas, then this should not be a big problem.

E) Basic areas will now be evaluated using the F.T.C.

F) Assignment: P. 286 {1 - 18}

G) An accumulator function is of the form: h(x) = $\int_a^x f(t)dt$

H) Evaluating accumulator functions from functions and graphs.

I) Using the calculator to graph an accumulator function.

J) Using the anti-derivative to derive an accumulator function.

K) The other version of the F.T.C. for constant “a”: $D_x[\int_a^x f(t)dt] = f(x)$

L) One interpretation of this version of the F.T.C. is being on a ladder, going up a step and then heading back down a step. You end up where you started.

M) We must use the chain rule on the other F.T.C. if the upper limit of integration is a function of x. $D_x[\int_a^{g(x)} f(t)dt] =$ g ’(x)*f(g(x))

N) An even more general version is: $D_x[\int_{h(x)}^{g(x)} f(t)dt] =$ g ’(x)*f(g(x)) - h ’(x)*f(h(x))

O) Interpreting aspects of the graph of an accumulator function when given a graph like problems #53 and #55.

P) Assignment: P. 287 {37 - 42, 48, 54, and 60}

5.5. TRAPEZOIDAL RULE

A) A trapezoid is not a rectangle, but it approximates the area very nicely.

B) The trapezoid rule is TRAP = $\frac{1}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n) * \Delta X$ where $\Delta X = \frac{b-a}{n}$

C) Given the concavity, determine whether we are overestimating or underestimating on our four area measures.

D) It doesn’t take much to show that TRAP = (LRAM+RRAM)/2

E) We should also be able to evaluate areas using trapezoids when the intervals are irregular – not the same length.

F) Volumes can be found by multiplying the area by the depth as in problem #7.

G) Examples using functions and tables. (Even using tables without numerical values).

H) We do not have to worry about Simpson’s Rule, but it uses little parabolas.

SIMP = $\frac{1}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n) * \Delta X$ where $\Delta X = \frac{b-a}{n}$

I) Assignment: P. 295 {1 - 6, 8}

Mandatory Review: P. 298 {1 - 6, 8, 9, 11 - 14, 15 - 36 multiples of 3, 38 - 42, 45, 46, 54, 56}
1997 BC Multiple Choice: {1, 9, 19, 22, 25, 85}