CHAPTER 8
L'HOPITAL'S RULE, IMPROPER INTEGRALS, AND PARTIAL FRACTIONS

8.1. L'HOPITAL'S RULE

A) Definition: An *indeterminate form* is a meaningless expression such as:

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad \infty\bullet 0,\quad \infty-\infty,\quad 1^{\infty},\quad 0^0,\quad \infty^0$$

B) Recall from chapter two that when we plugged into a function and got $\frac{0}{0}$ that we removed the discontinuity and plugged in again. There is another way.

C) L'Hopital's Rule: If f(a) = g(a) = 0 and f and g are differentiable on an open interval I containing a, and if g'(x) $\neq$ 0 on I as long as x $\neq$ a. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

D) We must make sure that the conditions are met before applying L'Hopital's Rule. We must be careful to avoid vertical asymptotes. Also note that we may have to apply the rule more than one time to find the limit.

E) The theorem works for a = ∞ (as x goes to infinity) as well as when $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

F) Recall that we can use the graph to confirm the limit.

G) For limits of the form $\infty - \infty$, the key is a common denominator.

H) Assignment: P. 423 {1 - 15, skip 3, 7, and 13}

I) The sneaky substitution for the form $\infty\bullet 0$: let h = $\frac{1}{x}$, then $\lim_{x\to\infty} x = \lim_{h\to 0^+}\frac{1}{h}$

J) To address 1^{∞}, 0^0, and ∞^0; recall that the limit of an exponent is the exponent of the limit. In this case we will introduce a natural log to help us evaluate the limit.

The procedure: $\lim_{x\to a} f(x) = \lim_{x\to a} e^{(\ln(f(x)))} = e^{\lim_{x\to a}(\ln(f(x)))}$

K) Your calculator may know how to evaluate limits.

L) Assignment: P. 424 {16 - 40 even, 49}

8.2. RELATIVE RATES OF GROWTH

A) Consider the following functions: 2^x, x^2, $\log_{\frac{1}{2}} x$, 2x, $x^{\frac{1}{2}}$, $\frac{1}{2}^x$, $\log_2 x$, $\frac{1}{2}$x, and x^x. How can we order their rates of growth? (We are interested in who wins as x gets large).

B) Let us define an analytical method to compare the rates of growth of two functions.

Defn: Let f(x) and g(x) be positive for x sufficiently large:

I) function f grows faster than function g as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty, \text{ or, equivalently, if } \lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$$

II) functions f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0$$

C) L'Hopital will help us order the functions from part A.

D) Consider what is taking place for the graph of $h(x) = \dfrac{f(x)}{g(x)}$.

E) Hints and techniques:

I) To compare two exponential functions, write (the ratio of two numbers) raised to the x power. If the ratio is greater than one, the beast goes to infinity. If the ratio is between zero and one then the beast heads to zero.

II) To compare power function with the square root of an expression, re-write the power function as the square root of its square. Recall that the limit of a square root is the square root of a limit. Note that the function you choose for the numerator will determine if you need to use L'Hopital.

F) Order and Oh-Notation.

Defn: Let f(x) and g(x) be positive for x sufficiently large:

I) If f grows at a slower rate than g, then f = o(g)

II) If f grows at the same rate or a slower rate than g, then f = O(g)

Note: If f grows at a faster rate, then g grows at a slower rate, and g = o(f)

G) Assignment: P.431 {2 - 30 even, 31 - 34}

8.3. IMPROPER INTEGRALS

A) Defn: Integrals with infinite limits of integration or infinite discontinuities are called *improper integrals*.

B) Improper integrals either converge to a number, or diverge to $\pm$ infinity.

C) We take care of improper integrals with infinite limits of integration as follows:

I) If one of the limits of integration is infinite, we assign it a letter (say, a) and then take $\lim_{a \to \infty}$ of the integral with a replacing the infinity.

II) If both limits of integration are infinite, we break the integral into two pieces, each having only one infinite limit of integration.

D) We take care of improper integrals involving vertical asymptotes as follows:

I) If the vertical asymptote is one of the limits of integration, we assign it a letter and then take the limit as the variable approaches that letter from the appropriate side.

II) If a vertical asymptote exists within the interval, we break the integral into two pieces each having only one limit of integration needing to be approached by a certain side.

E) Objects with infinite lengths and finite areas or infinite surfaces and finite volumes.

F) In order to evaluate the improper integrals within this section you will have to recall a couple of items because the book decided to get nasty:

I) The forms of the inverse trig functions, and the limits of these functions at infinity.

II) Sneaky substitutions and integration by parts.

G) Assignment: P. 442 {2 - 12 even, 18, 25, 26}

H) Convergence Testing compares the behavior of a function that we do not know with one in which we are familiar.

I) Let us become familiar with $\frac{1}{x^p}$, p > 0:

I) Find all p such that $\int_1^\infty \frac{1}{x^p}\,dx$ will converge.

II) Find all p such that $\int_0^1 \frac{1}{x^p}\,dx$ will converge.

J) Let us become familiar with $\frac{1}{e^x}$ and prove that $\int_1^\infty \frac{1}{e^x}\,dx$ converges.

K) The Direct Comparison Test: Let f and g be continuous on [a, ∞) with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

I) If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges

II) If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges

III) If $\int_a^\infty g(x)dx$ diverges, then no conclusion can be drawn for $\int_a^\infty f(x)dx$

IV) If $\int_a^\infty f(x)dx$ converges, then no conclusion can be drawn for $\int_a^\infty g(x)dx$

L) The Direct Comparison Test is a lot of writing for something that is so visually intuitive.

M) The Limit Comparison Test: If the positive functions f and g are continuous on [a, ∞)

and if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$, $0 < L < \infty$, then

$\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ either both converge or both diverge.

N) Final thoughts on convergence testing:

I) If you can evaluate the integral and it is a finite number, the integral converges.

II) If you can evaluate the integral and it is $\pm$ infinity, the integral diverges.

III) If you are looking at one of the tests, make sure that you meet the conditions of the test. The graph is very helpful here, the algebra is not too tedious.

IV) If you are looking at one of the tests, make sure that you are comparing the function to a function whose convergence is known.

V) If you want to compare to a function containing a sin or cos, consider the function without that term (or something close to it) as the comparison function.

O) Assignment: P. 442 {1, 7, 9, 30 - 44 even}

8.4. PARTIAL FRACTIONS AND INTEGRAL TABLES

A) Partial fractions allow you to take a rational function whose denominator has factors and write it as the sum of different rational functions. It is often easier to integrate a rational function that has been broken down by partial fractions.

B) The technique of partial fractions can only be used when the degree of the numerator is less than the degree of the denominator. If this is not the case use long division to make it proper.

C) The basic technique of partial fractions for linear factors (x - p)(x - q) in the denominator:

- I) Set the original rational expression equal to terms with each factor in the denominator with constants for each numerator.
- II) Multiply each side by the LCD
- III) Set the coefficients on each side equal to one another and solve for the numerator constants.

D) An alternate technique for partial fractions of non-repeated factors and a calculator note.

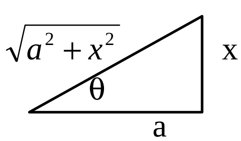
E) Other partial fractions to consider but will not be on the A.P. exam:

- I) Repeated linear factors.
- II) Quadratic factors.

F) Tables of integration exist for many types of integrals. See appendix #7 on p.628.

G) Recall the notes about tricky integrals from section two of chapter six.

H) Trigonometric substitution changes terms into factors.

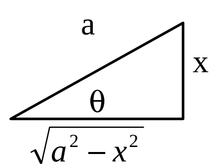


Let tan θ = $\frac{x}{a}$

then x = a tan θ

use sec θ

so $\sqrt{a^2 + x^2}$ = a sec θ

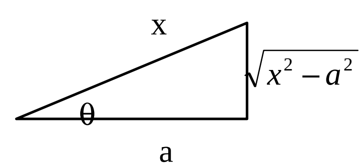


Let sin θ = $\frac{x}{a}$

then x = a sin θ

use cos θ

so $\sqrt{a^2 - x^2}$ = a cos θ



Let sec θ = $\frac{x}{a}$

then x = a sec θ

use tan θ

so $\sqrt{x^2 - a^2}$ = a tan θ

I) Don't forget to include dx in your integral, or to convert from θ back into x.

J) What do you do with a coefficient on x^2? What if you still need to use tables?

K) Assignment: P. 452 {8 - 26 even, 29, 30, 36 - 44 even}

Mandatory Assignment: P. 454 {3 - 60 multiples of 3, 61 - 65; skip 21, 27a, and 54}