# CHAPTER 9
# INFINITE SERIES

## 9.1. POWER SERIES

A) We have considered whether an integral converges or diverges, our next goal is to consider an infinite series and see if it converges or diverges. We will then use this information to help us write many types of functions as infinite polynomials.
B) Recall the geometric sequence and series and when an infinite geometric series converges.
C) Examples of infinite geometric series from economics and decimals.
D) Definition: An *alternating series* has consecutive terms with different signs.
E) Finding the nth term and the sum (if convergent) of various geometric series.
F) Note that not all series are geometric. Much of the rest of this chapter considers non-geometric series and whether they converge or diverge.
G) Assignment: P. 466 {1 - 7, 9 - 12, 15, 16, 27, 28, 38}

H) Here is where things get interesting. We can re-write an infinite polynomial as a rational expression if we let $a_1 = 1$ and $r = x$ within the infinite geometric series formula. Graphing each side shows that the graphs are not the same for all values of x. Where they do line up is called the interval of convergence. Note that the interval of convergence for a geometric series is consistent with the conditions of the geometric series formula.

I) Generalizing, we define: A *power series* centered at x = 0 is of the form $\sum_{n=0}^{\infty} c_n x^n$

Note that if the $c_n$ are geometric, then the series is geometric.

J) Definition: A power series centered at x = a is of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$

Graphically, we just shift the polynomial a units to the right.
K) By adjusting $a_1$ and r we can get different infinite polynomials to be equal to other rational expressions. Note again their intervals of convergence.
L) We can take known geometric power series and either differentiate or integrate them to create other, non geometric, power series! Not only that, the infinite polynomial could now be equal to natural logs and inverse tangent functions. You will have to trust that the intervals of convergence will not change except possibly now include the endpoints.
M) Assignment: P. 468 {42 - 52}

## 9.2. TAYLOR SERIES

A) We can construct our own power series for a function f(x). It is easiest to start with one that is centered at x = 0. Start with the power series:
$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + ...$
Find $P(0)$, $P'(0)$, $P''(0)$, $P'''(0)$, $P^{(4)}(0)$, . . . Discover the pattern.

B) To create our own power series for the function f(x), we must find the coefficients of P(x). The way we do that is set and solve:

$f(0) = 0!\bullet a_0$, $f'(0) = 1!\bullet a_1$, $f''(0) = 2!\bullet a_2$, $f'''(0) = 3!\bullet a_3$, $f^{(4)}(0) = 4!\bullet a_4, \ldots$

C) From part B we can generalize and say that $a_n = \dfrac{f^{(n)}(0)}{n!}$

D) Defn: A Taylor Series generated by f at x = 0 is

$$\frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

Note: When centered at zero it is also called a Maclaurin series.

E) Definition: A *Taylor series centered at x = a* is of the form:

$$\frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

F) Sometimes we only want the polynomial through the degree n term. We call it a Taylor polynomial of order n and it is denoted as, $P_n(x)$.

G) Generate the Maclaurin series on p. 477 and note their intervals of convergence.

H) We can manipulate the basic series to create polynomials for things like: Sin(2x), $xe^x$, . . .

I) Big Picture: Consider what we are doing from just one point. Sometimes we will be given values for the derivatives at a point instead of having to derive them as in #17.

J) Assignment: P.478 {1 - 4, 6 - 12, 14, 16, 21 a and b, 22}

## 9.3. TAYLOR'S THEOREM

A) Any Taylor series can be broken up into a Taylor polynomial of order n and the remaining terms. This is written as: $P(x) = P_n(x) + R_n(x)$. We are interested in the remainder because most of the time we must use the polynomial and would like to know how far we might be off. Note: Your calculator buttons generate Taylor polynomials.

B) The absolute value of the remainder is known as the truncation error.

C) On a geometric series, the remainder is a geometric series and can easily by evaluated. To get a handle on the remainder for any series we will consult Taylor's Theorem.

D) Taylor's Theorem: For a function f centered at x = a, there exists a value c between a and x such that $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.

E) The form of the remainder in Taylor's Theorem is often referred to as the Lagrange form of the error. While it may be difficult to find the value of c, it is often easy to find the boundary of the error. It is kind of like a butterfly in an aquarium. We may not know exactly where the butterfly is, but we know its boundaries.

F) How do we find the maximum error for a particular value of n over a given interval centered at x = a? If the remainder is strictly increasing or decreasing on the interval (which it usually is, if the interval is small), the maximum error will be at one of the end points. Here is a general guide:

1) Find $| R_n(x) |$ in terms of c.
2) Choose the largest value of x and evaluate $| R_n(x) |$ when c takes on values at the left

and the right side of the interval.

3) The largest value is the maximum that the error can be.

G) We can use the theorem to find the maximum error, or what values of x will fit into a certain error for a given order of polynomial.

H) For a sine or cosine function we have an option for our value of n.

I) Assignment: P. 486 {2 - 10 even, 11 - 15, 25 - 27}

## 9.4. RADIUS OF CONVERGENCE

A) We will begin the question of convergence of a Taylor series by just considering the convergence of an infinite series of numbers only.

B) Theorem 6: *The nth-Term Test for Divergence*:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \to \infty} a_n$ fails to exist or is different from zero.

C) Examples to consider: $\sum_{n=1}^{\infty} \frac{2n}{n+1}$, $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$, and $\sum_{n=1}^{\infty} \frac{1}{n}$.

D) Theorem 7: *The Direct Comparison Test* looks just like it did for areas in chapter 8.

E) Theorem 9: *The Ratio Test*: Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms, and with

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$. Then, (a) the series converges if $L < 1$, (b) the series diverges if $L > 1$ and (c) the test is inconclusive if $L = 1$

F) Assignment 1: P. 495 {1 - 15}

G) We are now ready to find the radius of convergence of a power series. As a prerequisite you must be convinced that for any value of x that $\lim_{n \to \infty} \frac{x^{n+1}}{(n+1)!} = 0$. Also note that when we take $\lim_{n \to \infty} |x|$ (which does not involve an n), we just get $|x|$.

H) If $|R_n(x)| \to 0$ as $n \to \infty$ for all x on some interval for a Taylor series, then the Taylor series converges to the function on the interval.

I) For f(x) = sin(x), we can show that the remainder goes to zero for all x.

J) Theorem 5: *The Convergence Theorem for Power Series*:

There are only three possibilities for a power series centered at x = a.

(a) The series converges only at x = a

(b) The series converges only within a radius about x = a (give or take endpoints)

(c) The series converges everywhere (the radius expands to infinity).

K) We use the ratio test to determine the radius of convergence. How? We just solve

$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

L) Note that If the degree of n is not the same in the numerator and denominator of the ratio, then you will either converge at a single point or for all real numbers.

M) Recall from Pre-Calculus how you solve inequalities of the form: $|\ (2x - 3)^2\ | < 1$.

N) To extend from the radius of convergence to the interval of convergence all we have to do is consider the endpoints. Section five is devoted to this idea.

O) Assignment: P. 495{20 - 34 do 3, skip 1}

## 9.5. TESTING CONVERGENCE AT ENDPOINTS

A) Definition: A *p-series* is of the form: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

B) Theorem 10, *The Integral Test:* If a sequence of positive terms, $\{a_n\}$ can be written as a function f(n), where f(x) is continuous, positive, and decreasing for all x ≥ some positive integer N; then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ either both converge or both diverge.

C) The Integral Test allows us to conclude that a p-series only converges for p>1.

D) Theorem 11: *The Limit Comparison Test*: Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N a positive Integer).

1. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c,\ 0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverges.

E) Theorem 12: *The Alternating Series Test*: The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ converges if all three of the following conditions are satisfied:

1. each $u_n$ is positive;
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N;
3. $\lim_{n \to \infty} u_n = 0$

F) Theorem 13: *The Alternating Series Estimation Theorem*: If an alternating series satisfies the conditions of theorem 12 then not only does it converge, but the truncation error is bounded by the next unused term.

G) Assignment: P. 506 {2 - 16 do 3, skip 1}

H) Now we can determine the interval of convergence by plugging in both end points. These are the values of x where the series converges conditionally. They are what we are most interested in.

I) A series is said to converge absolutely when the absolute value of the series converges. This does not happen as often because we no longer have the a.s.t. to prove convergence.

J) Now we can finish off the intervals of convergence for the memorizers of p. 477.

K) Assignment: P. 507 {28 - 42 part a only; do 3, skip 1} * Note that we already have the intervals of convergence from the previous section for many of these problems.
Mandatory Assignment: P. 509 {3 - 51 multiples of 3, 55 - 58, 60 - 63}