CHAPTER 10
PARAMETRIC, VECTOR, AND POLAR FUNCTIONS

10.1. PARAMETRIC FUNCTIONS

A) Recall that for parametric equations, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

B) If the equations x = f(t), and y = g(t) define y as a twice-differentiable function of x,

then at any point where $\frac{dx}{dt} \neq 0$, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

C) A parametric curve is differentiable at a point if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ both are differentiable at the point. A parametric curve is smooth if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are each continuous and not simultaneously zero.

D) Length (Arc Length) of a Smooth Parametrized Curve: If a smooth curve is traversed exactly once as t increases from a to b, the curve's length is L =

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

F) Assignment: P.518 {2 - 20 even, 24}

10.2. VECTORS IN THE PLANE - All of which was covered in Pre-Calculus

A) Definition: A vector in the plane is represented by a directed line segment.

B) Definition: Two vectors are equal if they have the same length and direction.

C) Definition: If **v** is a vector in the plane equal to the vector with initial point (0,0) and terminal point (v_1, v_2) then the component form of **v** is: **v** = $\langle v_1, v_2 \rangle$

D) The direction of **v** = $\langle v_1, v_2 \rangle$ is θ = $Tan^{-1}\left(\frac{v_2}{v_1}\right)$ (Possibly adjusting for quadrants).

E) The length (or magnitude or norm) of **v** = $\langle v_1, v_2 \rangle$, denoted | **v** | = $\sqrt{v_1^{\,2} + v_2^{\,2}}$.

F) Definition: A unit vector has a length of one. Any non-zero vector can be scaled to a unit vector by dividing it by its length.

G) Definition: The zero vector is **0** = $\langle 0,0 \rangle$. It has a length of zero and no direction.

H) You should be able to operate on vectors (add, subtract, scalar multiples), for vectors in graphical and/or component form.

I) Definition: The *Dot Product* of vectors **u** = $\langle u_1, u_2 \rangle$, and **v** = $\langle v_1, v_2 \rangle$ is the number
u•v = $u_1v_1 + u_2v_2$

J) Theorem: The angle between two vectors **u** and **v** is: θ = $\cos^{-1}\left(\dfrac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}\right)$

K) Definition: The slope of a vector **u** = $\langle u_1, u_2 \rangle$ is $\dfrac{u_2}{u_1}$

L) Two vectors are orthogonal/normal if their dot product is zero. For any non-zero vector there are two unit vectors and two unit normal vectors.

M) The airplane example for vector addition.

N) Assignment: P. 527 {2 - 26 even, 44, 45}

10.3. VECTOR-VALUED FUNCTIONS

A) The standard unit vectors are **i** = $\langle 1,0 \rangle$ and **j** = $\langle 0,1 \rangle$. Any vector, **v** = $\langle a,b \rangle$ = a**i** + b**j**.

B) A vector-valued function takes parametric equations and writes each of x = f(t) and y = g(t) as components of the vector **r**(t) = f(t)**i** + g(t)**j**. The graph is the path of the vector. (To actually graph a vector-valued, we just use parametric equations).

C) Most of the Calculus of vector-valued functions is taken care of componentially.
If **r**(t) = f(t)**i** + g(t)**j**, then

I) $\lim_{t \to c} \mathbf{r}(t) = \lim_{t \to c} f(t)\mathbf{i} + \lim_{t \to c} g(t)\mathbf{j}$.

II) $\dfrac{d\mathbf{r}}{dt} = \dfrac{df}{dt}\mathbf{i} + \dfrac{dg}{dt}\mathbf{j}$

III) $\int \mathbf{r}(t)dt = \int f(t)dt\ \mathbf{i} + \int g(t)dt\ \mathbf{j} + C$

IV) $\int_a^b \mathbf{r}(t)dt = \int_a^b f(t)dt\ \mathbf{i} + \int_a^b g(t)dt\ \mathbf{j}$

D) Definitions for particle movement for the function **r**(t) = f(t)**i** + g(t)**j**

I) **v**(t) = $\dfrac{d\mathbf{r}}{dt}$ is the particle's velocity vector and is tangent to the curve.

II) | **v**(t) |, the magnitude of v, is the particle's speed.

III) **a**(t) = $\dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$ is the particles acceleration vector.

IV) $\dfrac{\mathbf{v}}{|\mathbf{v}|}$, a unit vector, is the direction of motion

E) Differentiability and smoothiocity of vector-valued functions mirror parametric function.

F) Note that velocity = speed • direction = $|\mathbf{v}| \dfrac{\mathbf{v}}{|\mathbf{v}|}$

G) Assignment: P.537 {6 - 26 even, 27, 33}

10.4. MODELING PROJECTILE MOTION

A) Let g be the force of gravity. $g = 9.8\ m/s^2 = 32\ ft/s^2$. Since gravity is only a vertical component, $\mathbf{a}(t) = -g\mathbf{j}$

B) If v_0 is the initial velocity and θ is the launch angle, the initial velocity can be broken down into its component parts as $\mathbf{v}(0) = v_0 \cos\theta\ \mathbf{i} + v_0 \sin\theta\ \mathbf{j}$.

C) We can solve for $\mathbf{v}(t) = v_0 \cos\theta\ \mathbf{i} + (-gt + v_0 \sin\theta)\ \mathbf{j}$

D) If the initial position is $\mathbf{r}(0) = h\mathbf{i} + k\mathbf{j}$, then

$$\mathbf{r}(t) = (v_0 \cos\theta\ t + h)\mathbf{i} + \left(\frac{-1}{2}gt^2 + v_0 \sin\theta\ t + k\right)\mathbf{j}$$

E) The baseball problem and other problems.

F) If the initial position is the origin then here are three handy formulas:

1) Maximum height $= \dfrac{(v_0 \sin\theta)^2}{2g}$ 2) Flight time $= \dfrac{2v_0 \sin\theta}{g}$ 3) Range $= \dfrac{{v_0}^2}{g}\sin(2\theta)$

G) Assignment: P. 549 {1 - 7, 11, 12, 26}

10.5. POLAR COORDINATES AND POLAR GRAPHS - A Pre-Calculus Review

A) We use (x,y) in the Cartesian Coordinate system and (r, θ) in the Polar Coordinate system.

B) Degenerate cases of on variable in Cartesian and in Polar.

C) Converting between coordinate systems:

I) From Polar to Cartesian: $x = r\cos\theta$, and $y = r\sin\theta$

II) From Cartesian to Polar: $r^2 = x^2 + y^2$, and $\tan\theta = \dfrac{y}{x}$

D) When graphing, make sure your domain is large enough.

E) Assignment: P. 558 {2 - 6 even, 8 - 56 multiples of 4}

10.6. CALCULUS OF POLAR CURVES

A) We will be viewing lines in terms of x and y. First we will write the radius as a function of θ. Since $x = r\cos\theta$, and $y = r\sin\theta$, we will use $x = f(\theta)\cos\theta$, and $y = f(\theta)\sin\theta$.

As with parametric equations, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$

B) Using the slope equation to find horizontal and vertical asymptotes.

1) Potential horizontal asymptotes exist when the top = 0.
2) Potential vertical asymptotes when the bottom = 0.
3) Actual horizontal asymptotes exist when the top = 0 and the bottom $\neq 0$.
4) Actual vertical asymptotes exist when the bottom = 0 and the top $\neq 0$.
5) If the top and bottom are simultaneously zero, we must use L'Hopital to find the value of the slope. It may be a horizontal or vertical asymptote and it might be

something else.

6) We often want to write these tangent lines in Cartesian form and must use the parametric portion of x = f(θ) cos θ, or y = f(θ) sin θ to find the line.

C) Tangent lines that go through the origin (or pole) occur when the radius, f(θ) = 0. Once we find these values of θ, the tangent lines are of the form y = tan(θ) x, unless tan(θ) is undefined. In this case the line tangent to the pole is x = 0.

D) The good news is that the calculator can find $\frac{dy}{dx}$ (and $\frac{dr}{d\theta}$) for a given graph.

E) Assignment: p. 566 {2 - 12 do 3, skip 1}

F) Our first application of integration requires the area of a sector. A = $\frac{1}{2}\theta r^2$.

G) The area of the region between the origin and the curve r = f(θ), $\alpha \le \theta \le \beta$, is

$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

H) To find the area between two curves we must make sure that the radii are always positive and that the radii do not cross. The area of the region $0 \le r_1(\theta) \le r_2(\theta)$, $\alpha \le \theta \le \beta$, is

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) \, d\theta$$

I) The Length of a Polar Curve: if r = f(θ) has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point P(r, θ) traces the curve r = f(θ) exactly once as θ runs from α to β, then the length of the curve is: $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

J) Assignment: P. 566 {17, 18 - 27 multiples of 3, 33, 36, 40}

Mandatory Assignment: P. 569 {3 - 42 multiples of 3, 43 - 47, 51, 55, 58, 63 - 65, 67}