

# CHAPTER 11

## PARTIAL DERIVATIVES

### 1. FUNCTIONS OF SEVERAL VARIABLES

A) Definition: A function of two variables is a rule that assigns to each ordered pair of real numbers  $(x,y)$  in a set  $\mathbf{D}$  a unique real number denoted by  $f(x, y)$

\*The domain,  $\mathbf{D}$ , is the set of all  $(x,y)$

\* The range is  $\{f(x, y) | (x, y) \in \mathbf{D}\}$

B) As we move up a dimension, the notation is similar:

old:  $f(x) = x^2 + 1$  can be written as  $y = x^2 + 1$

new:  $f(x, y) = x^2 - y$  can be written as  $z = x^2 - y$

C) A brief discussion of independent and dependent variables

D) 3-D graphing on the calculator with animation

E) Definition: A level curve is a horizontal slice (parallel to the  $xy$  plane). it is of the form  $f(x, y) = k$

F) Definition: A contour map is a series of level curves graphed together

G) We need not stop with two variables in the domain. We can use as many as we want.

After a few it is easier to use vector notation

H) Assignment: P.756 {1, 5, 6, 7, 9, 10, 12, 13, 17, 29 - 37 odd}

### 2. LIMITS AND CONTINUITY

A) Definition: We write  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$  and say that the limit of  $f(x, y)$  as  $(x, y)$

approaches  $(a, b)$  is  $L$  if we can make the values of  $f(x, y)$  as close to  $L$  as we like by taking the point  $(x, y)$  sufficiently close to the point  $(a, b)$ , but not equal to  $(a, b)$

B) As we consider a two variable limit, we must approach the point from an infinite number of directions. If any two do not agree, then the limit does not exist.

C) Consider a limit by table and see that we are really only considering two/four directions

D) We can approach  $(0,0)$  by considering the lines of direction:  $x = 0$ ,  $y = 0$ , and  $y = mx$

E) Theorems we will use:

$$\lim_{(x,y) \rightarrow (a,b)} \quad \lim_{(x,y) \rightarrow (a,b)} \quad \lim_{(x,y) \rightarrow (a,b)}$$

$$x = a \quad y = b \quad c = c$$

The squeeze theorem also holds

F) The same basic rule applies to limits at a point: If you can plug it in, plug it in. If you cannot plug it in try to figure it out numerically, analytically, or graphically.

G) Continuity is an easy jump with more variables and makes intuitive sense.

H) Definiton: A function  $f$  of two variables is called continuous at  $(a, b)$  if:

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$$

I) Definiton: A function  $f$  is continuous on  $\mathbf{D}$  if  $f$  is continuous at every  $(a, b)$  in  $\mathbf{D}$

J) Assignment: P. 765 {3 - 33 multiples of 3}

### 3. PARTIAL DERIVATIVES

A) If we freeze one of the two variables in  $z = f(x, y)$ , then we can consider the rate of change of  $z$  with respect to the other variable within a plane. This is called a partial derivative

B) The partial derivative of  $f$  with respect to  $x$  at  $(a, b)$  is denoted  $f_x(a, b) = g'(a)$  where  $g(x) = f(x, b)$

C) Definition:  $f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$

D) Definition:  $f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$

E) There are various notations for partial derivatives

F) To find a partial derivative with respect to one variable, treat the other variables as constants

G) Compare and contrast partial derivatives and total derivatives by way of examples.

H) Investigate what the TI-89 does with multiple variables.

I) Functions in more than two variables work the same way: freeze all but the one variable you are interested in

J) The second partial derivative is a partial derivative of a partial derivative. We don't have to necessarily use the same variable on the second or higher derivative

K) Notation for higher order derivatives:  $(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$

L) Note the pictures on page 772

M) Clairaut's Theorem: Suppose  $f$  is defined on a disk  $D$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $d$ , then  $f_{xy}(a, b) = f_{yx}(a, b)$

N) Partial differential equations are equations involving partial derivatives. At this point we can but verify solutions

O) Can we produce the graphs of partial derivatives on our calculators?

P) Assignment: P. 775 {1, 2, 4, 5, 6, 9 - 54 multiples of 3, 67}

## 4. TANGENT PLANES AND LINEAR APPROXIMATIONS

- A) The tangent line goes 3-D becoming the tangent plane
- B) If we freeze any two distinct tangent lines through a point in 3-D, they determine a plane. This plane best approximates the surface at the given point.
- C) The equation of a plane is  $Ax + By + Cz - D = 0$ , let us graph a couple
- D) The point-slope equivalent of a plane at the point  $(x_0, y_0, z_0)$  is:  
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$
- E) In terms of partial derivatives, an equation of the tangent plane to the surface  $z = f(x, y)$  at the point  $(x_0, y_0, z_0)$  is:  $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
- F) As is the case in two dimensions, this plane is referred to as the linearization of the function at the given point. It is a good approximation for points close to  $x_0$  and  $y_0$
- G) A slightly different representation of this linearization is:  
$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
- H) Assignment: P. 787 {1 - 3, 9 - 15 odd}
- I) Definition: If  $z = f(x, y)$ , then  $f$  is differentiable at  $(a, b)$  if  $\Delta z$  can be expressed in the form  $\Delta z = f_x(a, b) \Delta x + f_y(a, b) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$  where  $\varepsilon_1$  and  $\varepsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$
- J) Theorem: If the partial derivatives of  $f_x$  and  $f_y$  exist near  $(a, b)$  and are continuous at  $(a, b)$ , then  $f$  is differentiable at  $(a, b)$
- K) The above theorem assures a “smooth surface”
- L) Upgrading differentials from two dimensions to three. Recall that  $dy = f'(x) dx$  and its graphical representation
- M) If  $z = f(x, y)$ , then  $dz = f_x(x, y)dx + f_y(x, y)dy$
- N) Additional dimensions add terms to the above
- O) A brief review/introduction to the cross product of two vectors
- P) Tangent planes to parametric surfaces
- Q) Assignment: P. 787 {20 - 28 even, 33, 36}

## 5. THE CHAIN RULE

- A) Recall the chain rule in two dimensions: If  $y = f(x)$  where  $x = g(t)$ , then  $\frac{dy}{dt} = \frac{df}{dx} \frac{dx}{dt}$
- B) The Chain Rule (Case 1) Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(t)$  and  $y = h(t)$  are both differentiable functions of  $t$ . Then  $z$  is a differentiable function of  $t$  and  $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$  or  $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$
- C) The Chain Rule (Case 2) Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(s, t)$  and  $y = h(s, t)$  are both differentiable functions of  $s$  and  $t$ . Then:  
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$
- D) The above may be generalized for more than two variables. The  $s$  and  $t$  are the independent variables;  $x$  and  $y$  are called intermediate variables, and  $z$  is the dependent variable.

E) Implicit Differentiation for  $F(x, y) = 0$  and  $y = f(x)$ :  $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$

F) If  $F(x, y, z) = 0$  and  $z = f(x, y)$ :  $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$  and  $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

G) Assignment: P. 796 {1 - 25 do one and skip 2, 27 - 33 odd}

## 6. DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

A) It is possible to find the derivative in a direction that is not parallel to the x or y axis. This is called the directional derivative.

B) Definition: The directional derivative of  $f$  at  $(x_0, y_0)$  in the direction of a unit vector  $\mathbf{u} = \langle a, b \rangle$  is  $D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$  if this limit exists.

C) Theorem: If  $f$  is a differentiable function of  $x$  and  $y$ , then  $f$  has a directional derivative in the direction of any unit vector  $\mathbf{u} = \langle a, b \rangle$  and  $D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$

D) Often times the directional derivative is given in terms of an positive angle with respect to the x axis. In this case  $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$  and  $D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos \theta + f_y(x, y)\sin \theta$

E) You may have noticed that the directional derivative can be written as the dot product:

$$\langle f_x(x, y), f_y(x, y) \rangle \bullet \mathbf{u}$$

F) Definition: The gradient of  $f$  is the vector  $\langle f_x(x, y), f_y(x, y) \rangle$ .

It is denoted **grad**  $f$  or  $\nabla f$

G) Another look at the directional derivative:  $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \bullet \mathbf{u}$

H) Adding dimensions means just adding another component to the directional derivative.

I) In which direction does the function  $f$  change the fastest and what is the maximum rate of change? The angle between two vectors is required to find the answer.

J) Theorem: Suppose  $f$  is a differentiable function of two or three variables. The maximum value of the directional derivative  $D_{\mathbf{u}}f(\mathbf{x})$  is  $|\nabla f(\mathbf{x})|$  and it occurs when  $\mathbf{u}$  has the same direction as the gradient vector  $\nabla f(\mathbf{x})$ .

K) Another look at the tangent plane to a level surface in terms of the gradient. The end result is that at the point  $(x_0, y_0, z_0)$ , the tangent plane to the level surface  $F(x, y, z) = k$  is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

L) Note the summary on page 807

M) Assignment: P. 808 {1, 3 - 36 multiples of 3}

## 7. MAXIMUM AND MINIMUM VALUES

A) Recall relative extrema on the plane. We will move up from curves on the plane to surfaces in three space with the use of partial differential equations

B) Definition: A function of two variables has a local maximum at a point  $(a, b)$  if  $f(a, b) \geq f(x, y)$  when  $(x, y)$  is contained in some disk with center  $(a, b)$ . Similarly,  $(a, b)$  is a local minimum if  $f(a, b) \leq f(x, y)$  when  $(x, y)$  is contained in some disk with center  $(a, b)$ .

C) Theorem: If  $f$  has a local maximum or minimum at  $(a, b)$  and the first-order partial derivatives of  $f$  exist there, then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$

D) Similar to critical points in two space, we define a critical point in three space as a point  $(a, b)$  where  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ , or at least one of these partial derivatives does not exist

E) Locating a critical point is sometimes easier than figuring out what type of extrema may or may not be present. It is possible to have relative maximums, relative minimums, or saddle points - a three dimensional type point of inflection

F) The second derivative test: Suppose the second partial derivatives of  $f$  are continuous on a disk with center  $(a, b)$ , and suppose that  $(a, b)$  is a critical point.

$$\text{Let } D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

a) If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local minimum.

b) If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local maximum.

c) If  $D < 0$ , then  $f(a, b)$  is not a local maximum or minimum.

G) We can now construct functions that we want to maximize or minimize with multiple variables. As in two space, examples include distance, surface area, volume, . . .

H) Recall the Extreme Value Theorem in two space: If  $f$  is continuous on the closed interval  $[a, b]$ , then  $f$  has both a maximum value and a minimum value on the interval.

I) The Extreme Value Theorem for Functions of Two Variables: If  $f$  is continuous on a closed, bounded set  $D$  in  $\mathbf{R}^2$ , then  $f$  attains an absolute maximum value  $f(x_1, y_1)$  and an absolute minimum value  $f(x_2, y_2)$  at some points  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $D$ .

J) How do you plug in the endpoints of the bounded set? This is difficult unless the extremities are easy to compute.

K) As in two space, to find the extreme values of a function, find the value of the function at the critical points within the domain and at the edges of the domain. From these choose the absolute maximum and the absolute minimum point.

L) Assignment: P. 818 {1, 3 - 15 multiples of 3, 23, 25, 31 - 37 odd}

## 8. LAGRANGE MULTIPLIERS

A) Goal: optimize a function  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$ . Physically, this happens when the surface represented by the function just touches the constraint surface. At this point the two surfaces share gradients that are scalars of one another. The scalar we will use is the greek letter lambda:  $\lambda$

B) To find the maximum and minimum values of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$  (if any exist):

- 1) Find all values of  $x, y, z$ , and  $\lambda$  such that  $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$   
and  $g(x, y, z) = k$
- 2) Evaluate  $f$  at all points found above to find the extreme values