CHAPTER 12
MULTIPLE INTEGRALS

1. DOUBLE INTEGRALS OVER RECTANGLES

A) We used the definite integral to find the area under a curve. Adding a dimension allows us to find the volume under a surface.

B) We use the idea of the Riemann sum to evaluate $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\Delta x = \int_a^b f(x)dx$

C) The first thing that changes when we up a dimension is that we have to sum the volumes of rectangular solids. The volume of a prism is the area of the base times the height. We have to get every square unit in the domain times the height. The base units are just $\Delta x \bullet \Delta y = \Delta A$ and the height is the surface for some point in the domain of that base area.

D) The volume under a surface is approximated by: $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$

E) Letting the number of intervals go to infinity in each direction of the base yields the following Riemann sum and its integral equivalent:

$$\lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A = \int_R \int f(x,y)dA$$

F) If f(x, y) $\geq$ 0, then the volume of the solid that lies above the rectangle R and below the surfce z = f (x, y) is: $V = \int_R \int f(x,y)dA$

G) Let us numerically compute some volumes using the midpoint or other corners of each base.

H) We can update the average value of a function in two variables. If A(R) is the area of the base region then the average value of the surface is: $f_{ave} = \frac{1}{A(R)} \int_R \int f(x,y)dA$

I) Assignment: P. 848 {1 - 11 odd }

2. ITERATED INTEGRALS

A) The big trick to multiple integration is performing partial integration from the inside out.

B) Fubini's Theorem: If f is continuous on the rectangle R = { (x, y) | a $\leq$ x $\leq$ b, c $\leq$ y $\leq$ d},

then $\int_R \int f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$

C) If the multiple integrand can be written as two independant functions then the multiple integral can be evaluated at the product of two integrals:

$$\int_a^b \int_c^d g(x) \bullet h(y)dydx = \int_a^b g(x)dx \bullet \int_c^d h(y)dy$$

D) Multiple integrals are easily computed in sections or altogether on the calculator

E) Assignment: P. 854 {3 - 30 multiples of 3}

3. DOUBLE INTEGRALS OVER GENERAL REGIONS

A) How do we move from rectangular base regions to generalized base regions? Let us first consider the differences between rectangular areas and generalized areas.

B) The end result for volumes under surfaces is that one of our set of limits of integration will be written functionally.

C) Type I: The base region has y as a function of x: If f is continuous and

$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

D) Type I: The base region has x as a function of y: If f is continuous and

$D = \{(x, y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$, then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

E) Sometimes a problem becomes possible by exchanging the limits of integration

F) Properties of multiple integration:

1) factor out a scalar
2) split up a sum or difference
3) the double integral of 1 is the area of the base region

G) A double integral version of the squeeze theorem

H) Assignment: P. 862 {3 - 36 multiples of 3}

4. DOUBLE INTEGRALS IN POLAR COORDINATES

A) Definition: A polar rectangle is an area of the plane described by:

$R = \{ (r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta \}$

B) The center of the polar rectangle along with the area of the sector of a circle are used to find $\Delta A = r(\Delta r)(\Delta \theta)$

C) The change to polar coordinates in a double integral: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

D) If r is a function of θ, then the base area can be described by a dynamic D: If f is continuous on a polar region of the form $D = \{ (r, \theta) \mid h_1(\theta) \le r \le h_2(\theta),\ \alpha \le \theta \le \beta \}$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

E) Examples are us

F) Assigment: P. 868 {1 - 6 all, 9 - 27 multiples of 3}

5. APPLICATIONS OF DOUBLE INTEGRALS

A)
B)
C)
D)
E)
F)
G) Assignment: P.

6. SURFACE AREA

A)
B)
C)
D)
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F)
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H)
I)
J)
K)
L) Assignment: P.

7. TRIPLE INTEGRALS

A)
B)
C)
D)
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F)
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I)
J)
K)
L) Assignment: P.

8. TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

A)
B)

C)
D)
E)
F)
G)
H)
I)
J)
K)
L) Assignment: P.

9. CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

A)
B)
C)
D)
E)
F)
G)
H)
I)
J)
K)
L) Assignment: P.